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THH and crystalline cohomology, I.
Drinfeld seminar.
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k perfect field of char. p.
A smooth k-algebra.

1) $THH(A)$.

2) Crystalline cohomology.

BMS recover crystalline cohomology from $THH(A)$. Analogous constructions for smooth $\mathcal{O}_{\mathbb{C}_p}$-algebras gives rise to A_{inf}-cohomology.

THH and HH

k a commutative ring.
A commutative k-algebra.

$$HH(A/k) \simeq A \otimes^{L}_{A \otimes_k A} A.$$

No 'op' because of commutativity. This is "the derived self-intersection of the diagonal".

$HH(A/k)$ takes values in $sCAlg_k$, or SCR_k. This is a simplicial model category.

We can view $A \in SCR_k$. Coproduct is the derived tensor product. Derived tensor product gives homotopy pushouts.

Thus, $HH(A/k)$ is a pushout

$$\begin{tikzcd} A \amalg A \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & HH(A/k). \end{tikzcd}$$

Now, SCR_k is tensored over spaces. Thus,

$$\boxed{HH(A/k) \simeq S^1 \otimes A} \longleftarrow \text{McClure-Schwänzl-Vogt.}$$

in SCR_k.

Thus, there is an S^1-action on $HH(A/k)$ and $HH(A/k)$ is universal for this property, with $A \to HH(A/k)$. Specifically, if $B \in SCR_k^{BS^1}$, then

$$\mathrm{Map}_{SCR_k}(A, B) \simeq \mathrm{Map}_{SCR_k^{BS^1}}(S^1 \otimes A, B).$$

$$S^1 \otimes A \cong HH(A/k)$$

We also see that S^1 acts on $HH(A/k) \in D(k)$. Thus, $HH(A/k)$ is naturally a dg-module over $C_\cdot(S^1, k)$.

$$C_\cdot(S^1,k) \cong k[\varepsilon]/(\varepsilon^2), \quad |\varepsilon| = 1.$$

Have to be careful about the product in non-zero char.

We get for any $C_\cdot(S^1, k)$-module V an operation

$$\varepsilon : H_*(V) \to H_{*+1}(V),$$

which is a differential. So,

$$\varepsilon : HH_*(A/k) \to HH_{*+1}(A/k)$$

is a differential. In fact, ε is a derivation.

Moreover, $HH_0(A) \cong A$ (natural). But, there is a universal example. That is, if A is a commutative k-algebra, then $\Omega^\bullet_{A/k}$ is the universal cdga with a map from A in degree 0. Hence, there is a nat. map of cdgas

$$(\Omega^*_{A/k}, d) \to (HH_*(A/k), \varepsilon).$$

This is a 1-categorical universal property. It says that the left adjoint to the forgetful functor from cdga$_{/k}$ to rings$/k$ given by $C^\bullet \mapsto C^0$ is $A \mapsto \Omega^\bullet_{A/k}$. We just freely adjoin differentials.

This is an degreewise iso. when A/k is smooth. In that case $\Omega^i_{A/k} \cong HH_i(A/k)$.

We can take S^1-fixed points: $HC^-(A/k) := HH(A/k)^{hS^1}$.

$$\begin{tikzcd} D(k) \arrow[r, bend left, "triv"] & D(k)^{BS^1} \arrow[l, bend left, "hS^1\text{-fixed points}"] \end{tikzcd}$$

Ex. k with S^1-action.

$$k^{hS^1} \simeq C^\cdot(\mathbb{CP}^\infty, k). \quad \text{Fix } x \in H^2(\mathbb{CP}^\infty, k).$$

$$X^{hS^1} \simeq RHom_{C_\cdot(S^1,k)}(k, X),$$

viewing $X \in D(k)^{BS^1}$ as a module over $C_\cdot(S^1, k)$.

We get $x : X^{hS^1}[-2] \to X^{hS^1}$.

The cofiber is X. This is classical for $HC^-(-/k)$.

Another construction.

$$X^{tS^1} \simeq X^{hS^1}[1/x].$$

Ex. $\pi_* k^{tS^1} \simeq k[x^{\pm 1}]$.

Rem. Probably, we are using that we are over $\mathbb{Z}$ here. Is this true for Sp^{BS^1}?

I guess not. But, the Tate construction is always given by killing some objects. Or, rather, it is the cofiber of the colimit preserving approximation to $(-)^{hS^1}$.

In any case, I'm not sure that Sp^{BS^1} will be unipotent. Certainly $ is not complex oriented.

Def. $HP(A/k) := HH(A/k)^{tS^1}$.

This is a module over k^{tS^1} and hence is 2-periodic on homology.

Thm. A/k smooth, k char. 0.

$$HP_0(A/k) \cong \bigoplus_{i \text{ even}} H^i(\Omega^\bullet_{A/k}, d).$$

$$HP_1(A/k) \cong \bigoplus_{i \text{ odd}} H^i(\Omega^\bullet_{A/k}, d).$$

Char. p.

k perfect.

char $k = p > 0$.

Now, use THH. Replace $D(k)$ with Sp. Replace SCR_k with $\mathbb{E}_\infty$-algebras, $\mathbb{E}_\infty$-ring spectra.

Background on THH...

Toy model: work over $\mathbb{Z}$.

Ex. $HH(k/\mathbb{Z}) \simeq k\langle\sigma\rangle$, divided power algebra on σ, $|\sigma| = 2$.

Rem. SCRs always have divided power structures on homology.

Thm (Bökstedt). $\pi_* THH(k) \cong k[\sigma]$, polynomial ring.

R a commutative k-algebra.

THH(R). S^1-action gives

1) $\varepsilon: \mathrm{THH}_i(R) \longrightarrow \mathrm{THH}_{i+1}(R)$

$\varepsilon^2 = 0$.

2) $\mathrm{THH}(k) \longrightarrow \mathrm{THH}(R)$ $\mathbb{E}_\infty$.

$\sigma \in \pi_2 \mathrm{THH}(k)$.

$\mathrm{THH}_0(R) \cong R$.

We get $(\Omega^*_{R/k}, d) \longrightarrow (\mathrm{THH}_*(R), \varepsilon)$. We also have $\underset{|\sigma|=2}{(k[\sigma], 0)} \longrightarrow (\mathrm{THH}_*(R), \varepsilon)$. So, we get

$$\left(\Omega^*_{R/k} \otimes_k k[\sigma], d\right) \longrightarrow (\mathrm{THH}_*(R), \varepsilon),$$

a map of complexes.

Thm (Hesselholt). If R/k is smooth, the map above is a degree-wise isomorphism.

Rem. There is a cofiber sequence

$$\mathrm{THH}(R)[2] \xrightarrow{\sigma} \mathrm{THH}(R) \longrightarrow \mathrm{HH}(R/k).$$

This is true even for R associative.

Use $\mathrm{THH}(R) \otimes_{\mathrm{THH}(k)} k \simeq \mathrm{THH}(R/k)$.

Akhil points out an easy way to prove this. Consider killing σ. On the LHS we get $\Omega^*_{R/k}$ and on the RHS we get get $\mathrm{THH}_*(R)/\sigma$. Also, $\pi_*(\mathrm{THH}(R)/\sigma) \cong \mathrm{HH}_*(R/k)$. Hence, by HKR, we are done if we know that THH(R) is σ-torsion free. This is true if R lifts to the sphere. See Ex. 3.4 of Mathew

"One-parameter deformation,"

on Kaledin. I guess we can then reduce to $\mathrm{THH}(\mathbb{F}_p[x^{\pm 1}])$.

Def. $TC^-(R) := THH(R)^{hS^1}$,

$TP(R) := THH(R)^{tS^1}$.

Ex. $\pi_* TC^-(k) \cong W(k)[\sigma, x]/(\sigma x - p)$,

$|\sigma| = 2$,

$|x| = -2$.

Rem. This seems to work over $\mathbb{Z}$ too, but the ring is not as nice.

Rem. $p = 0$ in $THH(k)$, but not equivariantly.
This is why $p \neq 0$ in $THH(k)^{hS^1}$.

Rem. $\pi_* TP(k) \cong W(k)[x^{\pm 1}]$.

$$\pi_* HP(k/\mathbb{Z}) \cong \left\{ \begin{array}{l} W(k) \text{ and add} \\ \text{divided powers freely} \\ \text{for the ideal } (p) \end{array} \right\} [x^{\pm 1}].$$

This ring is

$$W(k) \otimes_{\mathbb{Z}_p[y]} \underbrace{\Gamma(y)}_{\text{divided power algebra ring}}$$

There is p-torsion here.

For a k-algebra R, we get

$$TC^-(R)[-2] \longrightarrow TC^-(R) \longrightarrow THH(R).$$

Analog of Connes – Tsygan.

Also, $\quad TC^-(R)[\tfrac{1}{x}] \simeq TP(R).$

<u>Observation</u> (Bhatt). $\quad TP(R)/p \simeq \left(THH(R)^{tS^1}\right)/p \overset{\text{since } x \text{ is a unit}}{\simeq} \left(THH(R)^{tS^1}\right)/\sigma$

$p = x\sigma$

$$\simeq \left(THH(R)/\sigma\right)^{tS^1}$$
$$\simeq HH(R/k)^{tS^1}$$
$$\simeq HP(R/k).$$

So, we should think of $TP(R)$ as a kind of noncommutative crystalline cohomology.

— de Rham cohomology —

R/k smooth

k char $p>0$, perfect.

$(\Omega^\bullet_{R/k}, d)$

$H^*(\Omega^\bullet_{R/k}, d)$ is much bigger in char. p.

<u>Ex</u>. $H^0 \cong R^p$.

Cartier iso.

$$H^*(\Omega^{\bullet}_{R/k}, d) \underset{C^{-1}}{\simeq} \Omega^*_{R^{(1)}/k}.$$

Frobenius twist.

$R \otimes_{k} \phi k$.

$C^{-1}(x) = x^p$

$C^{-1}(dx) = x^{p-1}dx$

If x is invertible,

$$C^{-1}\left(\frac{dx}{x}\right) = \frac{dx}{x}.$$

Use local coordinates.

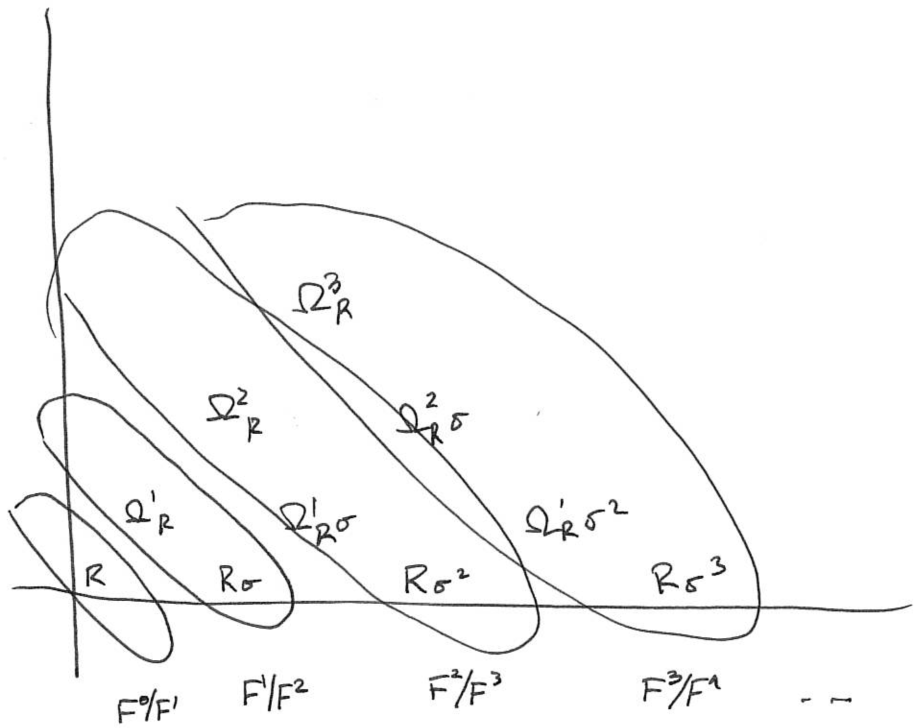
Main theorem today

Thm (Bhatt-Morrow-Scholze). R/k smooth k-algebra.
There is a functorial descending, multiplicative filtration

$$\{F^i THH(R)\}_{i \geq 0}.$$

$F^0 THH(R)$. $F^i \to F^{i-1}$. Associated graded is naturally equivalent to

$$F^i THH(R) / F^{i+1} THH(R) \simeq \left(\tau^{[0,i]}(\Omega^*_{R^{(1)}/k}, d)\right)[2i].$$



The associated graded pieces have homology given by differential forms. Compare Cartier isomorphisms.

<u>Rem</u>. Analog over $\mathcal{O}_{\mathbb{C}}$ produces a new cohomology theory.

<u>Thm</u> (BMS). There is a filtration $F^i TP(R)$, $i \in \mathbb{Z}$.

$$F^i/F^{i+1} \simeq \text{crys}(R)[2i].$$

<u>Rem</u>. There is also a filtration on $TC^-(R)$ with

$$F^i/F^{i+1}\,TC^-(R) \simeq \left(F^i \text{crys}(R)\right)[2i]$$

Nygaard filtration.